14.4 Expected Value

Objectives

1. Understand the meaning of expected value.
2. Calculate the expected value of lotteries and games of chance.
3. Use expected value to solve applied problems.

Life and Health Insurers' Profits Skyrocket 213% . . .*

How do insurance companies make so much money? When you buy car insurance, you are playing a sort of mathematical game with the insurance company. You are betting that you are going to have an accident—the insurance company is betting that you won't. Similarly, with health insurance, you are betting that you will be sick—the insurance company is betting that you will stay well. With life insurance, you are betting that, . . . well, . . . you get the idea.

Expected Value

Casinos also amass their vast profits by relying on this same mathematical theory—called **expected value**, which we will introduce to you in this section. Expected value uses probability to compare alternatives to help us make decisions.

Because of an increase in theft on campus, your school now offers personal property insurance that covers items such as laptops, iPods, cell phones, and even books. Although

*According to a report by Weiss Ratings, Inc., a provider of independent ratings of financial institutions.

the premium seems a little high, the insurance will fully replace any lost or stolen items. Our first example will help you get an idea of how probability can help you understand situations such as this.

EXAMPLE 1 *Evaluating an Insurance Policy*

Suppose that you want to insure a laptop computer, an iPhone, a trail bike, and your textbooks. Table 14.8 lists the values of these items and the probabilities that these items will be stolen over the next year.

a) Predict what the insurance company can expect to pay in claims on your policy.

b) Is $100 a fair premium for this policy?

Item	Value	Probability of Being Stolen	Expected Payout by Insurance Company
Laptop	$2,000	0.02	0.02($2,000) = $40
iPhone	$400	0.03	0.03($400) = $12
Trail bike	$600	0.01	0.01($600) = $6
Textbooks	$800	0.04	0.04($800) = $32

TABLE 14.8 Value of personal items and the probability of their being stolen.

SOLUTION:

a) From Table 14.8 the company has a 2% chance of having to pay you $2,000, or, another way to look at this is the company expects to lose on average 0.02($2,000) = $40 by insuring your computer. Similarly, the expected loss on insuring your iPhone is 0.03($400) = $12. To estimate, on average, what it would cost the company to insure all four items, we compute the following sum:

probability of iPhone being stolen; cost of iPhone; probability of books being stolen; cost of books

$$0.02(\$2{,}000) + 0.03(\$400) + 0.01(\$600) + 0.04(\$800) = \$90.$$

probability of computer being stolen; cost of computer; probability of bike being stolen; cost of bike

The $90 represents, *on average*, what the company can expect to pay out on a policy such as yours.

b) The $90 in part a) is telling us that if the insurance company were to write one million policies like this, it would expect to pay 1,000,000 × ($90) = $90,000,000 in claims. If the company is to make a profit, it must charge more than $90 as a premium, so it seems like a $100 premium is reasonable. 13

Quiz Yourself 13

In Example 1, if you were to drop coverage on your iPhone and add coverage on your saxophone that cost $1,400, what would the insurance company now expect to pay out in claims if the probability of the saxophone being stolen is 4% and the probability of your books being stolen is reduced to 3%?

The amount of $90 we found in Example 1 is called the *expected value* of the claims paid by the insurance company. We will now give the formal definition of this notion.

DEFINITION Assume that an experiment has outcomes numbered 1 to n with probabilities $P_1, P_2, P_3, \ldots, P_n$. Assume that each outcome has a numerical value associated with it and these are labeled $V_1, V_2, V_3, \ldots, V_n$. The **expected value** of the experiment is

$$(P_1 \cdot V_1) + (P_2 \cdot V_2) + (P_3 \cdot V_3) + \cdots + (P_n \cdot V_n).$$

In Example 1, the probabilities were $P_1 = 0.02$, $P_2 = 0.03$, $P_3 = 0.01$, and $P_4 = 0.04$. The values were $V_1 = 2{,}000$, $V_2 = 400$, $V_3 = 600$, and $V_4 = 800$.

Some Good Advice

Pay careful attention to what notation tells you to do in performing a calculation. In calculating expected value, you are told to *first* multiply the probability of each outcome by its value and *then* add these products together.

Expected Value of Games of Chance

EXAMPLE 2 Computing Expected Value When Flipping Coins

Number of Heads	Probability
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$

TABLE 14.9 Probabilities of obtaining a number of heads when flipping four coins.

What is the number of heads we can expect when we flip four fair coins?

SOLUTION: Recall that there are 16 ways to flip four coins. We will consider the outcomes for this experiment to be the different numbers of heads that could arise. Of course, these outcomes are not equally likely, as we indicate in Table 14.9. If you don't see this at first, you could draw a tree to show the 16 possible ways that four coins can be flipped. You would find that 1 of the 16 branches corresponds to no heads, 4 of the 16 branches would represent flipping exactly one head, and 6 of the 16 branches would represent flipping exactly two heads, and so on.

We calculate the expected number of heads by first multiplying each outcome by its probability and then adding these products, as follows:

probability of 0 heads; probability of 1 head

$$\left(\frac{1}{16}\cdot 0\right)+\left(\frac{4}{16}\cdot 1\right)+\left(\frac{6}{16}\cdot 2\right)+\left(\frac{4}{16}\cdot 3\right)+\left(\frac{1}{16}\cdot 4\right)=\frac{32}{16}=2$$

Thus we can expect to flip two heads when we flip four coins, which corresponds to our intuition.

We can use the notion of expected value to predict the likelihood of winning (or more likely losing) at games of chance such as blackjack, roulette, and even lotteries.

EXAMPLE 3 The Expected Value of a Roulette Wheel

Although there are many ways to bet on the 38 numbers of a roulette wheel,* one simple betting scheme is to place a bet, let's say \$1, on a single number. In this case, the casino pays you \$35 (you also keep your \$1 bet) if your number comes up and otherwise you lose the \$1. What is the expected value of this bet?

SOLUTION: We can think of this betting scheme as an experiment with two outcomes:

1. Your number comes up and the value to you is +\$35.
2. Your number doesn't come up and the value to you is −\$1.

Because there are 38 equally likely numbers that can occur, the probability of the first outcome is $\frac{1}{38}$ and the probability of the second is $\frac{37}{38}$. The expected value of this bet is therefore

probability of winning; amount won; probability of losing; amount lost

$$\left(\frac{1}{38}\cdot 35\right)+\left(\frac{37}{38}\cdot(-1)\right)=\frac{35-37}{38}=\frac{-2}{38}=-\frac{1}{19}=-0.0526.$$

*See Example 8 in Section 14.1 for a description of a roulette wheel.

This amount means that, on the average, the casino expects you to lose slightly more than 5 cents for every dollar you bet.

Now try Exercises 3 to 8.

The roulette wheel in Example 3 is an example of an unfair game.

DEFINITIONS If a game has an expected value of 0, then the game is called **fair**. A game in which the expected value is not 0 is called an **unfair game**.

Although it would seem that you would not want to play an unfair game, in order for a casino or a state lottery to make a profit, the game has to be favored against the player.

EXAMPLE 4 *Determining the Fair Price of a Lottery Ticket*

Assume that it costs $1 to play a state's daily number. The player chooses a three-digit number between 000 and 999, inclusive, and if the number is selected that day, then the player wins $500 (this means the player's profit is $500 − $1 = $499.)

a) What is the expected value of this game?

b) What should the price of a ticket be in order to make this game fair?

Outcome	Value	Probability
You win	$499	$\frac{1}{1{,}000}$
You lose	−$1	$\frac{999}{1{,}000}$

TABLE 14.10 Values and probabilities associated with playing the daily number.

SOLUTION:

a) There are 1,000 possible numbers that can be selected. One of these numbers is in your favor and the other 999 are against your winning. So, the probability of you winning is $\frac{1}{1{,}000}$ and the probability of you losing is $\frac{999}{1000}$. We summarize the values for this game with their associated probabilities in Table 14.10. The expected value of this game is therefore

$$\underbrace{\left(\frac{1}{1{,}000}\cdot 499\right)}_{\text{probability of winning; amount won}} + \underbrace{\left(\frac{999}{1{,}000}\cdot(-1)\right)}_{\text{probability of losing; amount lost}} = \frac{499-999}{1{,}000} = \frac{-500}{1{,}000} = -0.50.$$

This means that the player, on average, can expect to lose 50 cents per game. Notice that playing this lottery is 10 times as bad as playing a single number in roulette.

b) Let x be the price of a ticket for the lottery to be fair. Then if you win, your profit will be $500 - x$ and if you lose, your loss will be x. With this in mind, we will recalculate the expected value to get

$$\left(\frac{1}{1{,}000}\cdot(500-x)\right) + \left(\frac{999}{1{,}000}\cdot(-x)\right) = \frac{(500-x)-999x}{1{,}000} = \frac{500-1{,}000x}{1{,}000}.$$

We want the game to be fair, so we will set this expected value equal to zero and solve for x, as follows:

$$\frac{500-1{,}000x}{1{,}000} = 0$$

$$\cancel{1{,}000}\cdot\frac{500-1{,}000x}{\cancel{1{,}000}} = 1{,}000\cdot 0 = 0$$ Multiply both sides of the equation by 1,000.

$$500 - 1{,}000x = 0$$ Cancel 1,000 and simplify.

$$500 = 1{,}000x$$ Add 1,000x to both sides.

$$\frac{500}{1{,}000} = \frac{\cancel{1{,}000}x}{\cancel{1{,}000}} = x$$ Divide both sides by 1,000.

$$x = \frac{500}{1{,}000} = 0.50$$ Simplify.

HISTORICAL HIGHLIGHT

The History of Lotteries

Lotteries have existed since ancient times. The Roman emperor Nero gave slaves or villas as door prizes to guests attending his banquets, and Augustus Caesar used public lotteries to raise funds to repair Rome.

The first public lottery paying money prizes began in Florence, Italy, in the early 1500s; when Italy became consolidated in 1870, this lottery evolved into the Italian National Lottery. In this lottery, five numbers are drawn from 1 to 90. A winner who guesses all five numbers is paid at a ratio of 1,000,000 to 1. The number of possible ways to choose these five numbers is $C(90, 5) = 43{,}949{,}268$. Thus, as with most lotteries, these odds make the lottery a very good bet for the state and a poor one for the ordinary citizen.

Lotteries also played an important role in the early history of the United States. In 1612, King James I used lotteries to finance the Virginia Company to send colonists to the New World. Benjamin Franklin obtained money to buy cannons to defend Philadelphia, and George Washington built roads through the Cumberland mountains by raising money through another lottery he conducted. In fact, in 1776, the Continental Congress used a lottery to raise $10 million to finance the American Revolution.

This means that 50 cents would be a fair price for a ticket to play this lottery. Of course, such a lottery would make no money for the state, which is why most states charge $1 to play the game.

Now try Exercises 9 to 12.

Other Applications of Expected Value

Calculating expected value can help you decide what is the best strategy for answering questions on standardized tests such as the GMATs.

EXAMPLE 5 *Expected Value and Standardized Tests*

A student is taking a standardized test consisting of multiple-choice questions, each of which has five choices. The test taker earns 1 point for each correct answer; $\frac{1}{3}$ point is subtracted for each incorrect answer. Questions left blank neither receive nor lose points.

a) Find the expected value of randomly guessing an answer to a question. Interpret the meaning of this result for the student.

b) If you can eliminate one of the choices, is it wise to guess in this situation?

SOLUTION:

a) Because there are five choices, you have a probability of $\frac{1}{5}$ of guessing the correct result, and the value of this is +1 point. There is a $\frac{4}{5}$ probability of an incorrect guess, with an associated value of $-\frac{1}{3}$ point. The expected value is therefore

$$\left(\frac{1}{5}\cdot 1\right) + \left(\frac{4}{5}\cdot\left(-\frac{1}{3}\right)\right) = \frac{1}{5} + \frac{-4}{15} = \frac{3}{15} - \frac{4}{15} = -\frac{1}{15}.$$

Thus, you will be penalized for guessing and should not do so.

b) If you eliminate one of the choices and choose randomly from the remaining four choices, the probability of being correct is $\frac{1}{4}$ with a value of +1 point; the probability of being incorrect is $\frac{3}{4}$ with a value of $-\frac{1}{3}$. The expected value is now

$$\left(\frac{1}{4}\cdot 1\right)+\left(\frac{3}{4}\cdot\left(-\frac{1}{3}\right)\right)=\frac{1}{4}+\frac{-1}{4}=0.$$

You now neither benefit nor are penalized by guessing.

Now try Exercises 19 to 22. 14

Quiz Yourself 14

Calculate the expected value as in Example 5(b), but now assume that the student can eliminate two of the choices. Interpret this result.

Businesses have to be careful when ordering inventory. If they order too much, they will be stuck with a surplus and might take a loss. On the other hand, if they do not order enough, then they will have to turn customers away, losing profits.

EXAMPLE 6 *Using Expected Value in Business*

Cher, the manager of the U2 Coffee Shoppe, is deciding on how many of Bono's Bagels to order for tomorrow. According to her records, for the past 10 days the demand has been as follows:

Demand for Bagels	40	30
Number of Days with These Sales	4	6

She buys bagels for \$1.45 each and sells them for \$1.85. Unsold bagels are discarded. Find her expected value for her profit or loss if she orders 40 bagels for tomorrow morning.

SOLUTION: We can describe the expected value in words as

P(demand is 40) × (the profit or loss if demand is 40)
+ P(demand is 30) × (the profit or loss if demand is 30).

For 4 of the last 10 days the demand has been for 40 bagels, so $P(\text{demand is }40)=\frac{4}{10}=0.4$. Similarly, $P(\text{demand is }30)=\frac{6}{10}=0.6$.

We will now consider her potential profit or loss. If the demand is for 40 bagels, she will sell all of the bagels and make 40(\$1.85 − \$1.45) = 40(\$0.40) = \$16.00 profit. If the demand is for 30 bagels, she will make 30(\$0.40) = \$12.00 profit on the sold bagels, but will lose 10(\$1.45) = \$14.50 on the 10 unsold bagels. So, she will lose \$2.50. The following table summarizes our discussion so far if she orders 40 bagels.

Demand	Probability	Profit or Loss
40	0.4	\$16.00
30	0.6	−\$2.50

Quiz Yourself 15

Redo Example 6 assuming that Cher orders 30 bagels.

Thus, the expected value in profit or loss for ordering 40 bagels is

$$(0.40)(16)+(0.60)(-2.50)=+6.40+(-1.50)=4.90.$$

So she can expect a profit of \$4.90 if she orders 40 bagels. 15

Exercises 14.4

Looking Back*

These exercises follow the general outline of the topics presented in this section and will give you a good overview of the material that you have just studied.

1. How did we arrive at the term 0.02($2,000) in the equation that we wrote for the expected value in Example 1?
2. What did Examples 3 and 4 show us about the expected values of playing roulette versus playing the daily number?

Sharpening Your Skills

In Exercises 3 and 4, you are playing a game in which a single die is rolled. Calculate your expected value for each game. Is the game fair? (Assume that there is no cost to play the game.)

3. If an odd number comes up, you win the number of dollars showing on the die. If an even number comes up, you lose the number of dollars showing on the die.
4. You are playing a game in which a single die is rolled. If a four or five comes up, you win $2; otherwise, you lose $1.

In Exercises 5 and 6, you pay $1 to play a game in which a pair of fair dice are rolled. Calculate your expected value for the game. (Remember to subtract the cost of playing the game from your winnings.) Calculate the price of the game to make the game fair.

5. If a six, seven, or eight comes up, you win $5; if a two or 12 comes up, you win $3; otherwise, you lose the dollar you paid to play the game.
6. If a total lower than five comes up, you win $5; if a total greater than nine comes up, you win $2; otherwise, you lose the dollar you paid to play the game.

In Exercises 7 and 8, a card is drawn from a standard 52-card deck. Calculate your expected value for each game. You pay $5 to play the game, which must be subtracted from your winnings. Calculate the price of the game to make the game fair.

7. If a heart is drawn, you win $10; otherwise, you lose your $5.
8. If a face card is drawn, you win $20; otherwise, you lose your $5.

In Exercises 9–12, first calculate the expected value of the lottery. Determine whether the lottery is a fair game. If the game is not fair, determine a price for playing the game that would make it fair.

9. The Daily Number lottery costs $1 to play. You must pick three digits in order from 0 to 9 and duplicates are allowed. If you win, the prize is $600.
10. The Big Four lottery costs $1 to play. You must pick four digits in order from 0 to 9 and duplicates are allowed. If you win, the prize is $2,000.
11. Five hundred chances are sold at $5 apiece for a raffle. There is a grand prize of $500, two second prizes of $250, and five third prizes of $100.
12. One thousand chances are sold at $2 apiece for a raffle. There is a grand prize of $300, two second prizes of $100, and five third prizes of $25.
13. Grace Adler is planning to buy a franchise from Home Deco to sell decorations for the home. The table below shows average weekly profits, rounded to the nearest hundred, for a number of the current franchises. If she were to buy a franchise, what would her expected weekly profit be?

Average Weekly Profit	Number Who Earned This
$100	4
$200	8
$300	13
$400	21
$500	3
$600	1

14. For the past several years, the Metrodelphia Fire Department has been keeping track of the number of fire hydrants that have been opened illegally daily during heat waves. These data (rounded to the nearest ten) are given in the table below. Use this information to calculate how many hydrants the department should expect to be opened per day during the upcoming heat wave.

Hydrants Opened	Days
20	13
30	11
40	15
50	11
60	9
70	1

Applying What You've Learned

In Exercises 15–18, we describe several ways to bet on a roulette wheel. Calculate the expected value of each bet. We show a portion of a layout for betting on roulette in the diagram on page 702. When we say that a bet pays "k to 1," we mean that if a player wins, the player wins k *dollars as well as keeping his or her bet. When the player loses, he or she loses* $1. *Recall that there are 38 numbers on a roulette wheel.*

15. A player can "bet on a line" by placing a chip at location A in the figure. By placing the chip at A, the player is betting on 1, 2, 3, 0, and 00. This bet pays 6 to 1.

*Before doing these exercises, you may find it useful to review the note *How to Succeed at Mathematics* on page xix.

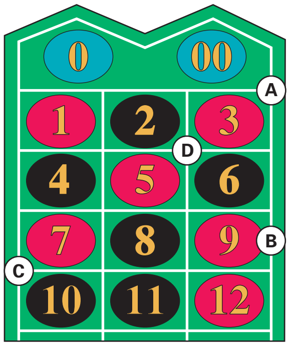

Figure for Exercises 15–18

16. A player can "bet on a square" by placing a chip at the intersection of two lines, as at location D. By placing a chip at D, the player is betting on 2, 3, 5, and 6. This bet pays 8 to 1.

17. A player can "bet on a street" by placing a chip on the table at location B. The player is now betting on 7, 8, and 9. This bet pays 8 to 1.

18. Another way to "bet on a line" is to place a chip at location C. By placing a chip at location C, the player is betting on 7, 8, 9, 10, 11, and 12. This bet pays 5 to 1.

In Exercises 19–22, a student is taking the GRE consisting of several multiple-choice questions. One point is awarded for each correct answer. Questions left blank neither receive nor lose points.

19. If there are four options for each question and the student is penalized $\frac{1}{4}$ point for each wrong answer, is it in the student's best interest to guess? Explain.

20. If there are three options for each question and the student is penalized $\frac{1}{3}$ point for each wrong answer, is it in the student's best interest to guess? Explain.

21. If there are five options for each question and the student is penalized $\frac{1}{2}$ point for each wrong answer, how many options must the student be able to rule out before the expected value of guessing is zero?

22. If there are four options for each question and the student is penalized $\frac{1}{2}$ point for each wrong answer, how many options must the student be able to rule out before the expected value of guessing is zero?

23. Assume that the probability of a 25-year-old male living to age 26, based on mortality tables, is 0.98. If a $1,000 one-year term life insurance policy on a 25-year-old male costs $27.50, what is its expected value?

24. Assume that the probability of a 22-year-old female living to age 23, based on mortality tables, is 0.995. If a $1,000 one-year term life insurance policy on a 22-year-old female costs $20.50, what is its expected value?

25. Your insurance company has a policy to insure personal property. Assume you have a laptop computer worth $2,200, and there is a 2% chance that the laptop will be lost or stolen during the next year. What would be a fair premium for the insurance? (We are assuming that the insurance company earns no profit.)

26. Assume that you have a used car worth $6,500 and you wish to insure it for full replacement value if it is stolen. If there is a 1% chance that the car will be stolen, what would be a fair premium for this insurance? (We are assuming that the insurance company earns no profit.)

27. A company estimates that it has a 60% chance of being successful in bidding on a $50,000 contract. If it costs $5,000 in consultant fees to prepare the bid, what is the expected gain or loss for the company if it decides to bid on this contract?

28. In Exercise 27, suppose that the company believes that it has a 40% chance to obtain a contract for $35,000. If it will cost $2,000 to prepare the bid, what is the expected gain or loss for the company if it decides to bid on this contract?

Communicating Mathematics

29. Explain in your own words the definition of expected value.

30. Explain in your own words how we determined the fair price of the game in Example 4. Why would the state not set this as the price to play the daily number?

Using Technology to Investigate Mathematics

31. Search the Internet for "the mathematics of lotteries." You might be surprised by the number of sites that you find selling software that can predict winning numbers. What is your reaction to these claims? Report on your findings.

32. Search the Internet for "expected value applets." Run some of these applets and report on your findings.

For Extra Credit

33. Nell's Bagels & Stuff, a local coffee shop, sells coffee, bagels, magazines, and newspapers. Nell has gathered information for the past 20 days regarding the demand for bagels. We list this information in the following table.

Demand for Bagels Sold	150	140	130	120
Number of Days with These Sales	3	6	5	6

Nell wants to use expected value to compute her best strategy for ordering bagels for the next week. She intends to order the same number each day and must order in multiples of 10; therefore, she will order either 120, 130, 140, or 150 bagels. She buys the bagels for 65 cents each and sells them for 90 cents.

a. What is Nell's expected daily profit if she orders 130 bagels per day? (*Hint:* First compute the profit Nell will earn if she can sell 150, 140, 130, and 120 bagels.)

b. What is Nell's expected daily profit if she orders 140 bagels per day?

34. Mike sells the *Town Crier*, a local paper, at his newsstand. Over the past 2 weeks, he has sold the following number of copies.

Number of Copies Sold	90	85	80	75
Number of Days with These Sales	2	3	4	1

Each copy of the paper costs him 40 cents, and he sells it for 60 cents. Assume that these data will be consistent in the future.

a. What is Mike's expected daily profit if he orders 80 copies per day? (*Hint:* First compute the profit Mike will earn if he can sell 90, 85, 80, and 75 papers.)

b. What is Mike's expected daily profit if he orders 85 copies per day?

35. a. Calculate the expected total if you roll a pair of standard dice.

b. Unusual dice, called *Sicherman dice*, are numbered as follows.

Red: 1, 2, 2, 3, 3, 4 Green: 1, 3, 4, 5, 6, 8

Calculate the expected total if you roll a pair of these dice.

36. Suppose we have two pairs of dice (these are called *Efron's dice*) numbered as follows.

Pair One: Red: 2, 2, 2, 2, 6, 6 Green: 5, 5, 6, 6, 6, 6

Pair Two: Red: 1, 1, 1, 5, 5, 5 Green: 4, 4, 4, 4, 12, 12

If you were to play a game in which the highest total wins when you roll, what is the better pair to play with?

37. In playing a lottery, a person might buy several chances in order to improve the likelihood (probability) of winning. Discuss whether this is the case. Does buying several chances change your expected value of the game? Explain.

38. A lottery in which you must choose 6 numbers correctly from 40 possible numbers is called a $\frac{40}{6}$ lottery. In general, an $\frac{m}{n}$ lottery is one in which you must correctly choose n numbers from m possible numbers. Investigate what kind of lotteries there are in your state. What is the probability of winning such a lottery?