Key

ICM Trigonometric Identities & Trig Equations Review Items

I. Reciprocal Identities

A) $\sin\theta = \frac{1}{\csc\theta}$ D) $\csc\theta = \frac{1}{\sin\theta}$

B) $\cos\theta = \frac{1}{\sec\theta}$ E) $\sec\theta = \frac{1}{\cos\theta}$

C) $\tan\theta = \frac{1}{\cot\theta}$ F) $\cot\theta = \frac{1}{\tan\theta}$

II. Quotient Identities

A) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

B) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

III. Pythagorean Identities

A) $\sin^2\theta + \cos^2\theta = 1$

B) $\tan^2\theta + 1 = \sec^2\theta$

C) $\cot^2\theta + 1 = \csc^2\theta$

Simplify each expression.

1. $\csc^2\theta - 1$

$\cot^2\theta$

2. $(1-\sin x)(1+\sin x)$

$1 - \sin^2 x$

$\cos^2 x$

3. $\sin^2\theta(\csc^2\theta - 1)$

$\frac{1}{\csc^2\theta}(\csc^2\theta - 1)$

$\frac{\csc^2\theta}{\csc^2\theta} - \frac{1}{\csc^2\theta} = 1 - \sin^2\theta$

$= \cos^2\theta$

4. $\cos\theta\tan\theta\csc\theta$

$\cos\theta \cdot \frac{\sin\theta}{\cos} \cdot \frac{1}{\sin\theta}$

1

5. $\frac{\csc\theta}{1+\cot^2\theta}$

$\frac{\csc\theta}{\csc^2\theta} = \frac{1}{\csc\theta}$

$= \sin\theta$

6. $\frac{1}{\sin^2\theta} - \frac{1}{\tan^2\theta}$

$\csc^2\theta - \cot^2\theta$

1

7. $\sin^2 x + \sin^2 x\cot^2 x$

$\sin^2 x (1 + \cot^2 x)$

$\sin^2 x \cdot \csc^2\theta$

$\sin^2 x \cdot \frac{1}{\sin^2 x} = 1$

Guidelines for Verifying Trig Identities

A. Begin with the most complicated side
B. Work on one side only
C. Rewrite sums or differences of quotients as one single quotient
D. Rewrite in terms of sine and cosine only
E. Factor (GCF or Difference of Squares)
F. Multiply (Foil or Distributive Property)

Verify each identity.

1. $\csc\theta\sin\theta-\sin^2\theta=\cos^2\theta$

LHS $\Rightarrow \csc\theta\cdot\frac{1}{\csc\theta}-\sin^2\theta$

$=1-\sin^2\theta=\cos^2\theta\Rightarrow$ RHS

2. $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

LHS $\Rightarrow \sin\theta\cot\theta+\sin\theta\tan\theta$

$=\frac{\sin\theta\cdot\cos\theta}{\sin\theta}+\sin\theta\cdot\frac{\sin\theta}{\cos\theta}$

$=\frac{\cos^2\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos\theta}=\frac{1}{\cos\theta}=\sec\theta$ RHS

3. $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=1$

LHS $\Rightarrow \csc^2\theta-\cot^2\theta=1\Rightarrow$ RHS

4. $\csc^4\theta-\csc^2\theta=\cot^4\theta+\cot^2\theta$

LHS $\Rightarrow \csc^2\theta(\csc^2\theta-1)$

$=(\cot^2\theta+1)\cot^2\theta$

$=\cot^4\theta+\cot^2\theta\Rightarrow$ RHS

5. $\csc x+\cot x=\frac{\sin x}{1-\cos x}$

LHS $\Rightarrow \frac{1}{\sin x}+\frac{\cos x}{\sin x}$

$\frac{1+\cos x}{\sin x}\cdot\frac{(1-\cos x)}{(1-\cos x)}=\frac{1-\cos^2 x}{(\sin x)(1-\cos x)}$

$=\frac{\sin^2 x}{\sin x(1-\cos x)}=\frac{\sin x}{1-\cos x}\Rightarrow$ RHS

6. $8\csc^2\theta-3\cot^2\theta=3+5\csc^2\theta$

LHS $\Rightarrow 8\csc^2\theta-3(\csc^2\theta-1)$

$=8\csc^2\theta-3\csc^2\theta+3$

$=3+5\csc^2\theta\Rightarrow$ RHS

7. $\frac{1}{1-\sec\theta}+\frac{1}{1+\sec\theta}=-2\cot^2\theta$

LHS $\Rightarrow \frac{1(1+\sec\theta)}{(1-\sec\theta)(1+\sec\theta)}+\frac{1(1-\sec\theta)}{(1-\sec\theta)(1+\sec\theta)}=\frac{2}{1-\sec^2\theta}=\frac{2}{-(\sec^2\theta-1)}=\frac{-2}{\tan^2\theta}$

$=-2\cot^2\theta\Rightarrow$ RHS

I. Sum and Difference Identities

A. $\cos(\alpha+\beta)=$

B. $\cos(\alpha-\beta)=$

C. $\sin(\alpha+\beta)=$

D. $\sin(\alpha-\beta)=$

E. $\tan(\alpha+\beta)=$

F. $\tan(\alpha-\beta)=$

Find the exact value of each expression.

1. $\cos\frac{\pi}{12}$ $= \cos 15$

$\cos(45-30) =$

$\cos 45\cos 30 + \sin 45\sin 30$

$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$

2. $\tan 105°$

$\tan(60+45)$

$\boxed{-2-\sqrt{3}}$

3. $\sin\frac{13\pi}{12}$

$= \sin 195$

$= \sin(150+45)$

$= \sin 150\cos 45 + \sin 45\cos 150$

$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{-\sqrt{3}}{2} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

~~4. $\csc(-15°)$~~ omit

5. $\sin 40°\cos 20° + \cos 40°\sin 20°$

$\sin(40+20)$

$\sin 60$

$\boxed{\frac{\sqrt{3}}{2}}$

6. $\cos 100°\cos 80° - \sin 100°\sin 80°$

$\cos 180$

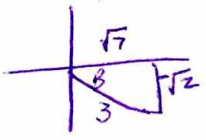
$\boxed{-1}$

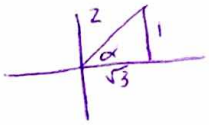
Find each of the following.

a) $\sin(\alpha+\beta)$ b) ~~$\tan(\alpha-\beta)$~~ omit

7. $\sin\alpha = \frac{1}{2},\ 0<\alpha<\frac{\pi}{2}$ $\cos\alpha = \frac{\sqrt{3}}{2}$

$\cos\beta = \frac{\sqrt{7}}{3},\ \frac{-\pi}{2}<\beta<0$ $\sin\beta = -\frac{\sqrt{2}}{3}$





a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$= \frac{1}{2}\cdot\frac{\sqrt{7}}{3} + \frac{-\sqrt{2}}{3}\cdot\frac{\sqrt{3}}{2}$

$= \boxed{\frac{\sqrt{7}-\sqrt{6}}{6}}$

~~b) $\tan(\alpha-\beta)$~~

8. Verify:

$\cos(\pi+\theta) = -\cos\theta$

LHS: $\cos\pi\cos\theta - \sin\pi\sin\theta$

$= -1\cos\theta - 0\cdot\sin\theta = -\cos\theta = RHS$

Trig Equations (I)

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(2) $\tan(60+45) = \dfrac{\sin 60\cos 45 + \sin 45\cos 60}{\cos 60\cos 45 - \sin 60 \sin 45}$

$= \dfrac{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}} = \dfrac{\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}}$

$= \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{2}-\sqrt{6}}$

$= \dfrac{(\sqrt{2}+\sqrt{6})(\sqrt{2}+\sqrt{6})}{(\sqrt{2}-\sqrt{6})(\sqrt{2}+\sqrt{6})}$

$= \dfrac{2+\sqrt{12}+\sqrt{12}+6}{2-6}$

$= \dfrac{8+2\sqrt{12}}{-4}$

$= \dfrac{8+4\sqrt{3}}{-4}$

$= \boxed{-2-\sqrt{3}}$

Solve each equation for $0 \le \theta < 2\pi$.

1. $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

2. $\cot\theta = \frac{\sqrt{3}}{3}$

$\tan\theta = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

3. $\cos\theta = \frac{-1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

* 4. $\sin(2\theta) = -\frac{\sqrt{2}}{2}$

$2\theta = \frac{5\pi}{4} \quad 2\theta = \frac{7\pi}{4}$

$\theta = \frac{5\pi}{8}$ $\theta = \frac{7\pi}{8}$

5. $2\tan\theta - 1 = 1$

$2\tan\theta = 2$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

6. $4\sin^2\theta - 3 = 0$

$\sin^2\theta = \frac{3}{4}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

* 7. $\csc\frac{3\theta}{2} = \frac{-2\sqrt{3}}{3}$

$\sin\frac{3\theta}{2} = \frac{3}{-2\sqrt{3}} = \frac{3\sqrt{3}}{-6} = -\frac{\sqrt{3}}{2}$

$\frac{3\theta}{2} = \frac{4\pi}{3}$ $\frac{3\theta}{2} = \frac{5\pi}{3}$

$\theta = \frac{8\pi}{9}$ $\theta = \frac{10\pi}{9}$

* 8. $\cot\left(\frac{\theta}{2} - \frac{\pi}{6}\right) = 1$

$\frac{\theta}{2} - \frac{\pi}{6} = \frac{\pi}{4}$

$\frac{\theta}{2} = \frac{5\pi}{12}$

$\theta = \frac{10\pi}{12}$

$= \frac{5\pi}{6}$

$\frac{\theta}{2} - \frac{\pi}{6} = \frac{5\pi}{4}$

$\frac{\theta}{2} = \frac{17\pi}{12}$

$\theta = \frac{34\pi}{12}$

$= \frac{17\pi}{6}$

I. Solving Trig Equations

A) Set trig function equal to a numerical value
B) Apply trig identities to rename expression in terms of one trig function
C) Factor:
   1. GCF
   2. Product of binomials
D) Divide $\cos\theta$ to produce $\tan\theta$

Solve for $0 \le \theta < 2\pi$.

1. $\cos^2\theta - \cos\theta = 0$

$\cos\theta(\cos\theta - 1) = 0$

$\cos\theta = 0$ or $\cos\theta - 1 = 0$

$\cos\theta = 1$

$0, \frac{\pi}{2}, \frac{3\pi}{2}$

2. $2\sin^2\theta - 3\sin\theta + 1 = 0$

$(2\sin\theta - 1)(\sin\theta - 1) = 0$

$2\sin\theta - 1 = 0$ $\quad \sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$ $\quad \sin\theta = 1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

3. $(\cot\theta - 1)(\csc\theta + 1) = 0$

$\cot\theta - 1 = 0$ or $\csc\theta + 1 = 0$

$\cot\theta = 1$ $\quad \csc\theta = -1$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\quad \sin\theta = -1$

$\frac{3\pi}{2}$

$\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}$

4. $2\cos^2\theta = \sin\theta + 1$

$2(1 - \sin^2\theta) = \sin\theta + 1$

$2 - 2\sin^2\theta = \sin\theta + 1$

$0 = 2\sin^2\theta + \sin\theta - 1$

$0 = (2\sin\theta - 1)(\sin\theta + 1)$

$2\sin\theta - 1 = 0$ $\quad \sin\theta + 1 = 0$

$\sin\theta = 1/2$ $\quad \sin\theta = -1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad \frac{3\pi}{2}$

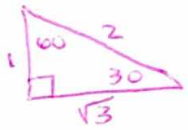
$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

5. $\sqrt{3}\sin\theta = \cos\theta$

$\frac{\sqrt{3}\sin\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}$

$\sqrt{3} = \cot\theta$

$\frac{1}{\sqrt{3}} = \tan\theta$



$\frac{\pi}{6}, \frac{7\pi}{6}$

6. $\sin 2\theta + \sin\theta = 0$

7. $\cos 2\theta = 1 - \sin\theta$

8. $\sin\theta\tan\theta = \sqrt{3}\sin\theta$